# §7 HERMITIAN LINE BUNDLES

 

As before, $M$ is a smooth manifold and $\pi : L \to M$ denotes a complex line bundle over $M$.

(7.1) DEFINITION: A HERMITIAN LINE BUNDLE is a line bundle for which every fibre $L_a$, $a \in M$, has a Hermitian metric depending smoothly on the points $l \in L$.

The Hermitian metric can be given by a map

$$H : L \times L \longrightarrow \mathbb{C}$$

such that $H|_{L_a \times L_a} : L_a \times L_a \to \mathbb{C}$ is a Hermitian scalar product for all $a \in M$. And the smoothness condition simply means that $H$ is smooth. We denote $H(l,l') = \langle l, l' \rangle$, $l, l' \in L$.

In case of the trivial line bundle $L = M \times \mathbb{C}$ we obtain a HERMITIAN METRIC $H_0$ by setting

$$H_0\big((a,z),(a,w)\big) = \big\langle (a,z),(a,w) \big\rangle := \bar{z}w \ , \ z, w \in \mathbb{C}, \ a \in M .$$

$H_0$ is called the CONSTANT Hermitian metric. We see that any other Hermitian metric on $L$ is given by a smooth $h : M \to \mathbb{R}$, $h(a) > 0$, for all $a \in M$ by

$$H(l,l') := h(a) \langle l, l' \rangle = h(a) H_0(l,l')$$

i.e.

$$H\big((a,z),(a,w)\big) = h(a)\,\bar{z}w .$$

(7.2) LEMMA: A hermitian line bundle $(L, H)$ such that the underlying line bundle is the trivial line bundle $M \times \mathbb{C}$ is isomorphic to the trivial line bundle with the constant Hermitian metric $H_0$.

□ Proof. The general case is of the form

$$H((a,z),(a,w)) = h(a)\bar{z}w,$$

and $\Phi : M \times \mathbb{C} \to M \times \mathbb{C}$, $(a,z) \mapsto (a, \sqrt[2]{h(a)}\,z)$,
defines an isomorphism of hermitian line bundles $(M \times L, H)$, $(M \times L, H_0)$ : $\Phi$ is an isomorphism of line bundles with $H(\ell,\ell') = H_0(\Phi(\ell),\Phi(\ell'))$, $\ell,\ell' \in M \times L$. □

From the local existence we can conclude that on every line bundle over a paracompact manifold there exists a Hermitian metric $H$ such that it is a Hermitian line bundle.

To a Hermitian line bundle $(L, H)$ one associates the circle bundle $L^1 \to M$, where

$$L^1 := \{ \ell \in L : H(\ell,\ell) = 1 \}.$$

This is a principal fibre bundle with the circle group $\mathbb{S}^1 = \{ z \in \mathbb{C} : |z| = 1 \}$ as its structure group. Conversely, if $P \to M$ is a principal fibre bundle with structure group $\mathbb{S}^1$ and $\varrho : \mathbb{S}^1 \to \mathbb{C}^\times = GL(1,\mathbb{C})$ is the natural representation $\varrho(z) = z : \mathbb{C} \to \mathbb{C}$, $w \mapsto zw$, then the associated vector bundle $L = P \times_{S^1} \mathbb{C}$ is a line bundle, where $S^1$ acts by scaler multiplication. The Hermitian metric $H$ on $L$ is then given by

$$H([x,z],[y,w]) := \bar{z}w, \quad \text{where } x,y \in P,\ z,w \in \mathbb{C}.$$

(7.3) PROPOSITION: The group of isomorphism classes of Hermitian line bundles $(L,H)$ over $M$ is isomorphic to $H^1(M,S^1) \cong H^2(M,\mathbb{Z})$.

Here, $H^1(M,S^1)$ is Čech cohomology with respect to the sheaf of $S^1$-valued smooth germs of functions on $M$ (see end of section 3).

We now study line bundles on which there exists a connection together with a Hermitian metric.

(7.4) DEFINITION: Given a Hermitian line bundle $(L,H)$ over $M$, a connection $\nabla$ on $L$ is called COMPATIBLE with $H$ if for all sections $s,t \in \Gamma(U,L)$ and all vector fields $X \in \mathfrak{V}(U)$, $U \subset M$ open, we have

$$L_X \langle s,t \rangle = \langle \nabla_X s, t \rangle + \langle s, \nabla_X t \rangle$$

$(\langle s,t \rangle = H(s,t))$.

Such a connection is also called a Hermitian connection.

(7.5) PROPOSITION: A connection $\nabla$ on $L$ is compatible with a Hermitian metric $H$, if and only if the local gauge potentials $(\alpha_j)_{j\in I}$ with respect to local trivializations $\varphi_j : L_{U_j} \longrightarrow U_j \times \mathbb{C}$, $\bigcup U_j = M$, can be chosen to be real one forms $\alpha_j \in \Omega^1(M,\mathbb{R})$.

☐ Proof. A collection of trivializations $\varphi_j : L_{U_j} \to U_j \times \mathbb{C}$ can by chosen in such a way that $\varphi_j$ is an isomorphism of hermitian line bundles with respect to the constant Hermitian metric $H_0$ on $U_j \times \mathbb{C}$, cf. Lemma (7.2). Now, $s,t \in \Gamma(U_j, L)$ have the form

$$s = f s_j\,,\quad t = g s_j\,,\quad g,f \in \mathcal{E}(U_j),$$

where $s_j(a) = \varphi_j^{-1}(a,1)$. Hence,

$$\langle s,t\rangle = \langle (a,f(a)),(a,g(a))\rangle_0 = \bar f(a) g(a)\,,\quad \text{i.e.}$$

$$L_X\langle s,t\rangle = (L_X\bar f)g + \bar f L_X g\,.$$

$$\langle \nabla_X s,t\rangle = \langle (a, L_X f(a) + 2\pi i \alpha_j(X) f(a)), (a,g(a))$$

$$= \left(\overline{L_X f} - 2\pi i\, \bar\alpha_j(\bar X)\, \bar f(a)\right) g(a)$$

$$\langle s,\nabla_X t\rangle = \bar f(a)\left(L_X g(a) + 2\pi i\, \alpha_j(X)\, g(a)\right).$$

Compatibality is therefore equivalent to

$$(L_X\bar f)g + \bar f L_X g = \overline{L_X f} + \bar f L_X g - 2\pi i\left(\bar\alpha_j(\bar X) - \alpha_j(X)\right)\bar f g\,.$$

If we restrict to real vector fields this equation for all $f, g$ amounts to

$$0 = \overline{\alpha_j}(X) - \alpha_j(X).$$

Hence $\alpha_j$ is a real one form.
The converse can be read off the above formulas. $\square$

Another characterization is the following:

(7.6) <u>PROPOSITION</u>:[*] A connection $\nabla$ on $L$ is compatible with a Hermitian metric $H$, if and only if for all non zero sections $s \in \Gamma(U, L^\times)$, $U \subset M$ open:

$$d\big(H(s,s)\big) = 2H(s,s)\,\mathrm{Re}\left(\frac{\nabla s}{s}\right),$$

where $\frac{\nabla s}{s}$ denotes the one form $\beta \in \Omega^1(U, \mathbb{C})$ given by

$$\nabla_X s = \beta(X)\, s, \quad X \in \mathcal{V}(U).$$

The compatibility is therefore also equivalent to

"If $s \in \Gamma(U, L^\times)$ is of length 1, i.e. $H(s,s) \equiv 1$,
then $\frac{\nabla s}{s}$ is purely imaginary"[*]

Concerning the existence of compatible connections we conclude:

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*Übung: Proof!

(7.7) COROLLARY: Let $(L, H)$ be a Hermitian line bundle. Then there exists a compatible connection $\nabla$. Given such a compatible connection $\nabla$ the set of all connections compatible with $H$ is the affine space

$$\nabla + 2\pi i\, \Omega^1(M, \mathbb{R}).$$

□ Proof. In the existence discussion in section 4 one only has to make sure to choose the one forms on the trivializations $L_{U_j} \to U_j \times \mathbb{C}$ as real one forms. The second statement follows from the description of all connections on $L$ as $\nabla + \Omega^1(M, \mathbb{C})$. □

A given $\nabla$ on the other hand may not be compatible to any metric $H$.

The curvature $\Omega = \mathrm{Curv}(\nabla, L)$ of a connection on $L$ compatible with a given Hermitian metric is always a real two form $\Omega \in \Omega^2(M, \mathbb{R})$.